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The Φ^4 quantum field in a scale invariant random metric

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Abstract
We discuss a D-dimensional Euclidean scalar field interacting with a scale invariant quantized metric. We assume that the metric depends on d-dimensional coordinates where $d < D$. We show that the interacting quantum fields have more regular short distance behaviour than the free fields. A model of a Gaussian metric is discussed in detail. In particular, in the Φ^4 theory in four dimensions we obtain explicit lower and upper bounds for each term of the perturbation series. It turns out that there is no charge renormalization in the Φ^4 model in four dimensions. We show that in a particular range of the scale dimension there are models in $D = 4$ without any divergencies.

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1. Introduction

We discuss a D-dimensional Euclidean scalar field interacting with a quantized scale invariant metric. The metric depends on d-dimensional coordinates. The simplest case, which arises from a Gaussian metric, will be discussed in detail. If $d = 2$ then we can give at least two non-Gaussian examples of such a scale invariant metric. In one example the metric is described as a two-dimensional $SL(D, R)$-valued field. As a second example we may consider the Polyakov model [1] in a non-critical dimension which in the conformal gauge is reduced to the two-dimensional Liouville model.

In the first model we treat a metric tensor on the Riemannian manifold (Euclidean formulation)

$$(G)^{AB} = g^{AB} \tag{1}$$

as a two-dimensional field G with values in a set of real symmetric positive definite $D \times D$ matrices G. We choose the metric in block diagonal form $G^{AB} = \delta^{AB}$ if $A, B > D - 2$ and for $A, B \leqslant D - 2$ the tensor $g^{\mu\nu}(\mathbf{x}_F)$ is a $(D-2) \times (D-2)$ matrix depending on $\mathbf{x}_F \in R^2$.

The manifold of positive definite matrices is homeomorphic to $R \times SL(D-2, R)/O(D-2)$. We choose a conformal invariant action for G

$$W(G) = \mathrm{Tr} \int \mathrm{d}\mathbf{x}_F G^{-1} \partial G G^{-1} \bar{\partial} G + \mathrm{WZW} \tag{2}$$

where $\partial = \partial_1 - \mathrm{i}\partial_2$ is the holomorphic derivative and WZW denotes the Wess–Zumino–Witten term [2]. The conformal invariance of this model has been shown in [3–5].

The first model is not invariant under the group of diffeomorphisms of the metric. In the second conformal invariant model of gravity we consider the string coordinates X^μ interacting with two-dimensional gravity in a way invariant under general coordinate transformations

$$W = \int \mathrm{d}\mathbf{x}_F \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X^\mu. \tag{3}$$

The classical model does not depend on the metric but the quantum one depends on the conformal Weyl factor. We choose $g_{ab} = \delta_{ab} \exp \chi$. A functional integral over X leads to the effective action

$$W_{\mathrm{eff}}(g) = \int \mathrm{d}\mathbf{x}_F (\partial_a \chi \partial_a \chi + \alpha \exp \chi). \tag{4}$$

This Liouville model is scale invariant [6, 7].

In sections 2 and 3 we discuss general scale invariant models. In section 4 we restrict ourselves to the Gaussian metric where we can obtain detailed estimates on the perturbation series of the $(\Phi\Phi^*)^2$ interaction (we discuss the complex scalar field instead of the real one just for the simplicity of the Gaussian combinatorics). It is shown that quantum fields interacting with a singular random metric are more regular than the free fields (a conjecture reviewed in [8]; see also [9]).

2. The scalar propagator

We consider a complex scalar matter field Φ in D dimensions interacting with gravitons depending only on a d-dimensional submanifold. We split the coordinates as $x = (\mathbf{x}_G, \mathbf{x}_F)$ with $\mathbf{x}_F \in R^d$. Without a self-interaction the $\Phi\Phi^*$ correlation function is equal to an average

$$\hbar \int \mathcal{D}g \exp\left(-\frac{1}{\hbar} W(G)\right) \mathcal{A}^{-1}(x, y) \tag{5}$$

over the gravitational field g of the Green function of the operator

$$-\mathcal{A} = \frac{1}{2} \sum_{\mu=1,\nu=1}^{D-d} g^{\mu\nu}(\mathbf{x}_F) \partial_\mu \partial_\nu + \frac{1}{2} \sum_{k=D-d+1}^{D} \partial_k^2. \tag{6}$$

We repeat some steps of [10] (our case here is simpler and more explicit). We represent the Green function by means of the proper time method

$$\mathcal{A}^{-1}(x, y) = \int_0^\infty \mathrm{d}\tau (\exp(-\tau\mathcal{A}))(x, y). \tag{7}$$

For a calculation of $(\exp(-\tau\mathcal{A}))(x, y)$ we apply the functional integral

$$K_\tau(x, y) = (\exp(-\tau\mathcal{A}))(x, y) = \int \mathcal{D}x \exp\left(-\frac{1}{2} \int \frac{\mathrm{d}\mathbf{x}_F}{\mathrm{d}t} \frac{\mathrm{d}\mathbf{x}_F}{\mathrm{d}t} \right.$$
$$\left. - \frac{1}{2} \int g^{\mu\nu}(\mathbf{x}_F) \frac{\mathrm{d}x_\mu}{\mathrm{d}t} \frac{\mathrm{d}x_\nu}{\mathrm{d}t}\right) \delta(x(0) - x) \delta(x(\tau) - y). \tag{8}$$

In the functional integral (8) we make a change of variables ($x \to b$) determined by Stratonovitch stochastic differential equations [11]

$$\mathrm{d}x^{\Omega}(s) = e_A^{\Omega}(x(s))\,\mathrm{d}b^A(s) \tag{9}$$

where for $\Omega = 1, 2, \ldots, D-d$

$$e_a^{\mu} e_a^{\nu} = g^{\mu\nu}$$

and $e_A^{\Omega} = \delta_A^{\Omega}$ if $\Omega > D - d$.

As a result of the transformation $x \to b$ the functional integral becomes Gaussian with the covariance

$$E[b_a(t)b_c(s)] = \delta_{ac}\min(s,t). \tag{10}$$

In contradistinction to [10] equation (9) can be solved explicitly. The solution q_τ of equation (9) consists of two vectors $(\mathbf{q}_G, \mathbf{q}_F)$ where

$$\mathbf{q}_F(\tau, \mathbf{x}_F) = \mathbf{x}_F + \mathbf{b}_F(\tau) \tag{11}$$

and $\mathbf{q}_G$ has the components (for $\mu = 1, \ldots, D-d$)

$$q^{\mu}(\tau, \mathbf{x}) = x^{\mu} + \int_0^{\tau} e_a^{\mu}(\mathbf{q}_F(s, \mathbf{x}_F))\,\mathrm{d}b^a(s). \tag{12}$$

The kernel is

$$\begin{aligned} K_\tau(x,y) &= E[\delta(y - q_\tau(x))] \\ &= E\left[\delta(\mathbf{y}_F - \mathbf{x}_F - \mathbf{b}_F(\tau))\prod_{\mu}\delta(y_\mu - q_\mu(\tau, x))\right]. \end{aligned} \tag{13}$$

Using equation (12) and the Fourier representation of the δ-function we write equation (13) in the form

$$\begin{aligned} K_\tau(x,y) = (2\pi)^{-D}\int \mathrm{d}\mathbf{p}_G\,\mathrm{d}\mathbf{p}_F\,E\bigg[\exp\bigg(&\mathrm{i}\mathbf{p}_F(\mathbf{y}_F - \mathbf{x}_F) + \mathrm{i}\mathbf{p}_G(\mathbf{y}_G - \mathbf{x}_G) \\ &- \mathrm{i}\mathbf{p}_F\,\mathbf{b}_F(\tau) - \mathrm{i}\int p_\mu e_a^{\mu}(\mathbf{q}(s, \mathbf{x}_F))\,\mathrm{d}b^a(s)\bigg)\bigg]. \end{aligned} \tag{14}$$

3. The scale invariant model

In general, we cannot calculate the average over the metric explicitly. However, the scale invariance of the metric is sufficient for a derivation of the short distance behaviour of the scalar propagator.

Let us note that $\sqrt{\tau}b(s/\tau) \simeq \tilde{b}(s)$, where $\tilde{b}$ denotes an equivalent Brownian motion (the equivalence means that both random variables have the same correlation functions). Then using the scale invariance of e with the index γ we can write

$$e(\sqrt{\tau}\mathbf{x}_F) \simeq \tau^{-\frac{\gamma}{2}}\tilde{e}(\mathbf{x}_F). \tag{15}$$

Hence, in equation (12)

$$q^{\mu}(\tau, \mathbf{x}) = x^{\mu} + \tau^{\frac{1}{2}-\frac{\gamma}{2}}\int_0^1 \tilde{e}_a^{\mu}\big(\tau^{-\frac{1}{2}}\mathbf{x}_F + \tilde{\mathbf{b}}_F(s)\big)\,\mathrm{d}\tilde{b}^a(s). \tag{16}$$

The expectation value over e is

$$\langle K_\tau(x,y)\rangle = \tau^{-\frac{D-d}{2}(1-\gamma)-\frac{d}{2}}\big\langle E\big[\delta\big((\mathbf{y}_F - \mathbf{x}_F)\tau^{-\frac{1}{2}} - \tilde{\mathbf{b}}_F(1)\big)\delta\big(\tau^{-\frac{1}{2}+\frac{\gamma}{2}}(y-x) - \eta\big)\big]\big\rangle \tag{17}$$

where

$$\eta^{\mu} = \int_0^1 \tilde{e}_a^{\mu}\big(\tau^{-\frac{1}{2}}\mathbf{x}_F + \tilde{\mathbf{b}}_F(s)\big)\,\mathrm{d}\tilde{b}^a(s).$$

Let $P(\mathbf{u}, \mathbf{v})$ be the joint distribution of $(\eta, \tilde{\mathbf{b}}_F(1))$ (P does not depend on $\mathbf{x}_F$ because of the translational invariance). Then the propagator of the Φ field is

$$\hbar\langle \mathcal{A}^{-1}(x, y)\rangle = \hbar \int_0^\infty \mathrm{d}\tau\, \tau^{-(1-\gamma)\frac{D-d}{2}-\frac{d}{2}} P\big((\mathbf{x}_G - \mathbf{y}_G)\tau^{(-1+\gamma)/2}, (\mathbf{x}_F - \mathbf{y}_F)\tau^{-\frac{1}{2}}\big). \tag{18}$$

Equation (18) in momentum space has the representation

$$\hbar\langle \mathcal{A}^{-1}(\mathbf{k}_G, \mathbf{k}_F)\rangle = \hbar \int_0^\infty \mathrm{d}\tau\, \tilde{P}\big(\tau^{\frac{1-\gamma}{2}}\mathbf{k}_G, \sqrt{\tau}\mathbf{k}_F\big) \tag{19}$$

where $\tilde{P}$ denotes the Fourier transform of P. Using equation (14) we may write

$$\hbar\langle \mathcal{A}^{-1}(\mathbf{k}_G, \mathbf{k}_F)\rangle = \hbar \int_0^\infty \mathrm{d}\tau \big\langle E\big[\exp \mathrm{i}\big(\sqrt{\tau}\mathbf{k}_F\tilde{\mathbf{b}}_F(1) + \tau^{\frac{1}{2}-\frac{\gamma}{2}}\mathbf{k}_G\eta_G\big)\big]\big\rangle.$$

The dispersion relation (relating the frequency to the wave number) is determined by (after an analytic continuation $k_0 \to \mathrm{i}k_0$)

$$(\langle \mathcal{A}^{-1}(\mathbf{k}_G, \mathbf{k}_F)\rangle)^{-1} = 0.$$

It can be concluded from equation (18) that in general the dispersion relation will be different from the standard one (resulting from a wave equation) $k_0 \sim |\mathbf{k}|$. In particular, we can see that if $|\mathbf{k}_F| \gg |\mathbf{k}_G|$ then $\langle \mathcal{A}^{-1}(\mathbf{k}_G, \mathbf{k}_F)\rangle \sim |\mathbf{k}_F|^{-2}$ whereas if $|\mathbf{k}_G| \gg |\mathbf{k}_F|$ then $\langle \mathcal{A}^{-1}(\mathbf{k}_G, \mathbf{k}_F)\rangle \sim |\mathbf{k}_G|^{-\frac{2}{1-\gamma}}$. In the configuration space, the propagator tends to infinity if both $|\mathbf{x}_F - \mathbf{y}_F|$ and $|\mathbf{x}_G - \mathbf{y}_G|$ tend to zero. However, the singularity depends in a rather complicated way on the approach to zero. It becomes simple if either $|\mathbf{x}_F - \mathbf{y}_F| = 0$ or $|\mathbf{x}_G - \mathbf{y}_G| = 0$. So, if $|\mathbf{x}_F - \mathbf{y}_F| = 0$ then we make a change of the time variable

$$\tau = t|\mathbf{x}_G - \mathbf{y}_G|^{\frac{2}{1-\gamma}}. \tag{20}$$

Using equation (18) we obtain the factor depending on $|\mathbf{x}_G - \mathbf{y}_G|$ in front of the integral and a bounded function A of coordinates, i.e.

$$\langle \mathcal{A}^{-1}(x, y)\rangle = A|\mathbf{x}_G - \mathbf{y}_G|^{-D+2}.$$

If $|\mathbf{x}_G - \mathbf{y}_G| = 0$ then we change the time variable

$$\tau = t|\mathbf{x}_F - \mathbf{y}_F|^2.$$

As a result

$$\langle \mathcal{A}^{-1}(x, y)\rangle = A|\mathbf{x}_F - \mathbf{y}_F|^{-(D-2)(1-\gamma)} \tag{21}$$

with a certain bounded function A. We can see that in the $\mathbf{x}_G$ coordinate the singularity remains unchanged but the propagator is more regular in the $\mathbf{x}_F$ coordinate.

It is not possible to calculate the probability distribution P exactly. Choosing as a first approximation $\eta \simeq \mathbf{b}_G(1)$ we obtain

$$P(\mathbf{u}, \mathbf{v}) = (2\pi)^{-\frac{D}{2}} \exp\left(-\frac{\mathbf{u}^2}{2} - \frac{\mathbf{v}^2}{2}\right).$$

In this approximation

$$\hbar\langle \mathcal{A}^{-1}(\mathbf{k}_G, \mathbf{k}_F)\rangle = \frac{\hbar}{2}\int_0^\infty \mathrm{d}\tau \exp\left(-\frac{1}{2}\tau^{1-\gamma}|\mathbf{k}_G|^2 - \frac{1}{2}\tau|\mathbf{k}_F|^2\right). \tag{22}$$

In the Higgs model we need the mass term $\int :\Phi\Phi^*:$. The perturbation series for the mass will be finite if the integral

$$\int_V \mathrm{d}\mathbf{x}_F\, \mathrm{d}\mathbf{x}_G |\langle \mathcal{A}^{-1}(x, y)\rangle|^2$$

is finite for any bounded region V. This integral is convergent if

$$\int_{|k|>\epsilon} \mathrm{d}\mathbf{k}_G\,\mathrm{d}\mathbf{k}_F |\langle \mathcal{A}^{-1}(k)\rangle|^2 < \infty \tag{23}$$

for any $\epsilon > 0$. It can be checked that if $D = 4$ then for any $\gamma > 0$ the integral (23) is convergent.

In fact, in D dimensions the convergence of the integral (23) (with the proper time representation (18)) follows from the convergence of the integral

$$\int_0^1 \mathrm{d}\tau_1 \int_0^1 \mathrm{d}\tau_2 (\tau_1 + \tau_2)^{-\frac{d}{2}} \left(\tau_1^{1-\gamma} + \tau_2^{1-\gamma}\right)^{-\frac{D-d}{2}}. \tag{24}$$

It is finite if $d + (1-\gamma)(D-d) < 4$. Together with the inequality $\gamma < \frac{1}{2}$ we obtain the inequality $D < 8 - d$.

The integral (24) comes directly from the approximation (22). However, we can prove this result, in general, under some mild regularity assumptions on the Fourier transform (19) of P defined in equation (18). So,

$$\int |\langle \mathcal{A}^{-1}(k)\rangle|^2\,\mathrm{d}k = \int \mathrm{d}\tau\,\mathrm{d}\tau' \tilde{P}\left(\tau^{\frac{1-\gamma}{2}}\mathbf{k}_G, \sqrt{\tau}\mathbf{k}_F\right)\tilde{P}\left(\tau'^{\frac{1-\gamma}{2}}\mathbf{k}_G, \sqrt{\tau'}\mathbf{k}_F\right)\mathrm{d}\mathbf{k}_G\,\mathrm{d}\mathbf{k}_F.$$

We introduce the spherical coordinates on the plane $\tau = r\cos\theta$, $\tau' = r\sin\theta$. Then, just by scaling r we derive the result (23) under the assumption that the integral over θ is finite.

4. The $(\Phi\Phi^*)^2$ model in a random Gaussian metric

After the general scale invariant models of gravity in the previous sections we now consider a Gaussian model. For the Gaussian model we can prove explicit upper and lower bounds on the correlation functions. We consider a complex scalar matter field Φ in D dimensions. Using equation (12) and the Fourier representation of the δ-function we write equation (14) in the form

$$K_\tau(x, y) = (2\pi)^{-D+d} \int \mathrm{d}\mathbf{p}_G \exp(\mathrm{i}\mathbf{p}_G(\mathbf{y}_G - \mathbf{x}_G)) \\ \times E\left[\delta(\mathbf{y}_F - \mathbf{x}_F - \mathbf{b}_F(\tau)) \exp\left(-\mathrm{i}\int p_\mu e^\mu_a(\mathbf{q}(s, \mathbf{x}_F))\,\mathrm{d}b^a(s)\right)\right].$$

The random variables $\mathbf{b}_F$ and b^a are independent. Hence, using the formula [11]

$$E\left[\exp \mathrm{i}\int f_a(\mathbf{q}_F)\,\mathrm{d}b^a\right] = E\left[\exp\left(-\frac{1}{2}\int f_a f_a\,\mathrm{d}s\right)\right]$$

we can rewrite equation (14) solely in terms of the metric tensor

$$K_\tau(x, y) = (2\pi)^{-D+d} \int \mathrm{d}\mathbf{p}_G \exp(\mathrm{i}\mathbf{p}_G(\mathbf{y}_G - \mathbf{x}_G)) \\ \times E\left[\delta(\mathbf{y}_F - \mathbf{x}_F - \mathbf{b}_F(\tau)) \exp\left(-\frac{1}{2}\int p_\mu p_\nu g^{\mu\nu}(\mathbf{q}(s, \mathbf{x}_F))\,\mathrm{d}s\right)\right]. \tag{25}$$

We assume that the metric $g^{\mu\nu}$ is Gaussian with the short distance correlations

$$\langle g^{\mu\nu}(\mathbf{x}_F) g^{\sigma\rho}(\mathbf{y}_F)\rangle = -D^{\mu\nu;\sigma\rho}(\mathbf{x}_F - \mathbf{y}_F) = -C^{\mu\nu;\sigma\rho}|\mathbf{y}_F - \mathbf{x}_F|^{-4\gamma} \tag{26}$$

where C is a scale invariant tensor and D must be positive definite if the momentum integrals in the final formula are to exist. Such a requirement contradicts the positive definiteness of the action for the gravitational field. However, in the Einstein gravity the conformal modes give a negative contribution to the action. The model of a conformally flat metric with $C^{\mu\nu;\sigma\rho} = \delta^{\mu\nu}\delta^{\sigma\rho}$ would be satisfactory for our purposes (a proper contour rotation in the complex space of metrics is needed in order to perform the functional integral; such an interpretation of the functional integral over the conformal modes has been considered also in quantum gravity [12, 13]). Then, the integral (17) over g can be calculated as

$$\langle K_\tau(x,y)\rangle = (2\pi)^{-D}\int \mathrm{d}\mathbf{p}_G \exp(\mathrm{i}\mathbf{p}_G(\mathbf{y}_G-\mathbf{x}_G))E\Bigg[\delta(\mathbf{y}_F-\mathbf{x}_F-\mathbf{b}_F(\tau)) \times \exp\left(-\frac{1}{4}\int_0^\tau p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}_F(s)-\mathbf{b}_F(s'))\,\mathrm{d}s\,\mathrm{d}s'\right)\Bigg]. \tag{27}$$

The propagator of equation (3) has the form

$$\langle \mathcal{A}^{-1}(x,y)\rangle = \int_0^\infty \mathrm{d}\tau\,\tau^{-\frac{d}{2}-(D-d)(1-\gamma)/2} F\big(\tau^{-\frac{1}{2}}(\mathbf{y}_F-\mathbf{x}_F), \tau^{-\frac{1}{2}+\frac{\gamma}{2}}(\mathbf{y}_G-\mathbf{x}_G)\big). \tag{28}$$

There is a restriction on the allowed singularity of the two-point function of the metric field. So, if the random variable in the exponential (25) is to be well defined (without any renormalization) then the expectation value of its square should be finite

$$\left\langle E\left[\left(\int_0^\tau g^{\mu\nu}(\mathbf{q}(s,\mathbf{x}_F))\,\mathrm{d}s\right)^2\right]\right\rangle < \infty.$$

This expectation value leads to the integral (if the singularity of D is $|\mathbf{u}_F|^{-4\gamma}$)

$$\int \mathrm{d}\mathbf{u}_F \int \mathrm{d}s \int_0^s \mathrm{d}s'(s-s')^{-\frac{d}{2}} \exp\left(-\frac{1}{2}\mathbf{u}_F^2\big/(s-s')\right)|\mathbf{u}_F|^{-4\gamma}.$$

The integral is finite if $\gamma < \frac{1}{2}$.

We calculate the expectation value in the gravitational field (26) of the product of any number of propagators. First, consider the two-point function of the Wick square which we need for the mass term in the Higgs model

$$\begin{aligned}\langle :\Phi\Phi^*:(x):\Phi\Phi^*:(y)\rangle &= \langle(\mathcal{A}^{-1}(x,y))^2\rangle \\ &= (2\pi)^{-2D+2d}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 \int \mathrm{d}\mathbf{p}_G\,\mathrm{d}\mathbf{p}'_G \exp(\mathrm{i}\mathbf{p}_G(\mathbf{y}_G-\mathbf{x}_G) \\ &\quad + \mathrm{i}\mathbf{p}'_G(\mathbf{y}_G-\mathbf{x}_G))E\Bigg[\delta(\mathbf{y}_F-\mathbf{x}_F-\mathbf{b}_F(\tau_1))\delta(\mathbf{y}_F-\mathbf{x}_F-\mathbf{b}'_F(\tau_2)) \\ &\quad \times \exp\bigg(-\frac{1}{4}\int_0^{\tau_1}\int_0^{\tau_1} p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}_F(s)-\mathbf{b}_F(s'))\mathrm{d}s\,\mathrm{d}s' \\ &\quad - \frac{1}{4}\int_0^{\tau_2}\int_0^{\tau_2} p'_\mu p'_\sigma p'_\nu p'_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}'_F(s)-\mathbf{b}'_F(s'))\,\mathrm{d}s\,\mathrm{d}s' \\ &\quad - \frac{1}{2}\int_0^{\tau_1}\int_0^{\tau_2} p_\mu p_\nu p'_\rho p'_\sigma D^{\mu\nu;\sigma\rho}(\mathbf{b}_F(s)-\mathbf{b}'_F(s'))\,\mathrm{d}s\,\mathrm{d}s'\bigg)\Bigg].\end{aligned} \tag{29}$$

We introduce the spherical coordinates on the (τ_1,τ_2)-plane where $\tau_1 = r\cos\theta$ and $\tau_2 = r\sin\theta$. Next, we rescale the momenta $\mathbf{k}_F = \mathbf{p}_F\sqrt{r}, \mathbf{k}'_F = \mathbf{p}'_F\sqrt{r}, \mathbf{k}_G = \mathbf{p}_G r^{\frac{1}{2}-\frac{\gamma}{2}}$ and $\mathbf{k}'_G = \mathbf{p}'_G r^{\frac{1}{2}-\frac{\gamma}{2}}$. Then, we can see that

$$\langle :\Phi\Phi^*:(x):\Phi\Phi^*:(y)\rangle = \int \mathrm{d}\theta\,\mathrm{d}r\,r r^{-d-(1-\gamma)(D-d)} F\big(\theta, r^{-\frac{1}{2}}(\mathbf{x}_F-\mathbf{y}_F), r^{-\frac{1}{2}+\frac{\gamma}{2}}(\mathbf{x}_G-\mathbf{y}_G)\big). \tag{30}$$

It follows just by scaling the coordinates (the r-integral scales as twice the τ-integral in equation (18)) that for short distances

$$\langle :\Phi\Phi^*:(x):\Phi\Phi^*:(y)\rangle \simeq (\langle\mathcal{A}^{-1}(x,y)\rangle)^2. \tag{31}$$

We need to prove that correlation functions (28) are finite and nonzero. We first show that the bilinear form $(f_j, \langle\mathcal{A}^{-1}\rangle f_l)$ is finite and nonzero on a dense set of functions f. For this purpose we choose

$$f_{\mathbf{k}}(\mathbf{x}_G) = (2\pi a)^{-\frac{D-d}{2}} \exp\left(-\frac{a}{2}\mathbf{x}_G^2 + \mathrm{i}\mathbf{k}\mathbf{x}_G\right).$$

Then,

$$\begin{aligned}(f_{\mathbf{k}}, \langle\mathcal{A}^{-1}\rangle f_{\mathbf{k}'}) = (2\pi)^{-D+d}\int_0^\infty \mathrm{d}\tau\,\tau^{-\frac{d}{2}}\int \mathrm{d}\mathbf{p}_G E\bigg[\delta\big(\tau^{-\frac{1}{2}}(\mathbf{y}_F-\mathbf{x})_F - \mathbf{b}_F(1)\big)\\ \times\exp\bigg(-\frac{1}{2a}(\mathbf{p}_G-\mathbf{k}^2) - \frac{1}{2a}(\mathbf{p}_G-\mathbf{k}')^2\\ -\frac{1}{4}\tau^{2-2\gamma}\int_0^1 p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}_F(s)-\mathbf{b}_F(s'))\,\mathrm{d}s\,\mathrm{d}s'\bigg)\bigg]. \qquad (32)\end{aligned}$$

Both sides depend on $\mathbf{x}_F$ and $\mathbf{y}_F$ because we integrated only $\mathbf{x}_G$ and $\mathbf{y}_G$. In our estimates we apply Jensen inequalities in the form (for real functions A and f)

$$E[\exp A] \geqslant \exp E[A] \qquad (33)$$

and

$$E\left[\exp\left(-\int_0^1 \mathrm{d}s\,\mathrm{d}s' f(s,s')\right)\right] \leqslant \int_0^1 \mathrm{d}s\,\mathrm{d}s' E[\exp(-f(s,s'))]. \qquad (34)$$

An upper bound can be obtained by means of the Jensen inequality (34) expressed in the form

$$\begin{aligned}(f_{\mathbf{k}}, \langle\mathcal{A}^{-1}\rangle f_{\mathbf{k}'}) \leqslant 2\int_0^\infty \mathrm{d}\tau\int_0^1 \mathrm{d}s\int_0^s \mathrm{d}s'\int \mathrm{d}\mathbf{u}_1\,\mathrm{d}\mathbf{u}_2\,\mathrm{d}\mathbf{p}_G\\ \times\tau^{-\frac{d}{2}}\exp\left(-\frac{1}{2a}(\mathbf{p}_G-\mathbf{k})^2 - \frac{1}{2a}(\mathbf{p}_G-\mathbf{k}')^2\right) p(s',\mathbf{u}_1)p(s-s',\mathbf{u}_2-\mathbf{u}_1)\\ \times p\big(1-s, \tau^{-\frac{1}{2}}(\mathbf{y}-\mathbf{x})-\mathbf{u}_2\big)\exp\left(-\frac{\tau^{2-2\gamma}}{4}p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}(\mathbf{u}_1-\mathbf{u}_2)\right)\end{aligned} \qquad (35)$$

where $p(s,\mathbf{u}) = (2\pi s)^{-\frac{d}{2}}\exp(-\mathbf{u}^2/2s)$. We can convince ourselves by means of explicit calculations (using a proper change of variables) that the integral on the rhs of equation (35) is finite. For the lower bound it will be useful to introduce the Brownian bridge starting from $\mathbf{x}$ and ending in $\mathbf{x}+\mathbf{u}$ [14] defined on the time interval [0, 1],

$$\mathbf{a}(\mathbf{x},\mathbf{u},s) = \mathbf{x} + \mathbf{u}s + \mathbf{c}(s) \qquad (36)$$

where $\mathbf{c}$ is the Gaussian process starting from 0 and ending at 0 with the correlation function

$$E[c_j(s')c_k(s)] = \delta_{jk}s'(1-s)$$

for $s' \leqslant s$. Then, the δ function in equation (27) defines the Brownian bridge and the Jensen inequality (33) takes the form

$$\begin{aligned}(f_{\mathbf{k}}, \langle\mathcal{A}^{-1}\rangle f_{\mathbf{k}'}) \geqslant (2\pi)^{-D+d}\int_0^\infty \mathrm{d}\tau\,\tau^{-\frac{d}{2}}\int \mathrm{d}\mathbf{p}_G\exp\bigg(-\frac{1}{2a}(\mathbf{p}_G-\mathbf{k})^2 - \frac{1}{2a}(\mathbf{p}_G-\mathbf{k}')^2\\ -\frac{1}{4}\tau^{2-2\gamma}\int_0^1 p_\mu p_\sigma p_\nu p_\rho E\big[D^{\mu\nu;\sigma\rho}\big(\mathbf{a}\big(0,\tau^{-\frac{1}{2}}\mathbf{y}_F-\tau^{-\frac{1}{2}}\mathbf{x}_F,s\big)\\ -\mathbf{a}\big(0,\tau^{-\frac{1}{2}}\mathbf{y}_F-\tau^{-\frac{1}{2}}\mathbf{x}_F,s'\big)\big)\big]\,\mathrm{d}s\,\mathrm{d}s'\bigg) \qquad (37)\end{aligned}$$

where the expectation value in the exponential on the rhs of equation (37) is equal to

$$\int \mathrm{d}\mathbf{u} \int \mathrm{d}s \int_0^s \mathrm{d}s' \omega(s,s')^{-\frac{d}{2}} \exp\left(-\frac{1}{2}\mathbf{u}^2/\omega(s,s')\right) \left|\mathbf{u} - \tau^{-\frac{1}{2}} s(\mathbf{y}_F - \mathbf{x}_F) + \tau^{-\frac{1}{2}} s'(\mathbf{y}_F - \mathbf{x}_F)\right|^{-4\gamma} \tag{38}$$

where $\omega(s,s') = (s-s')(1-s+s')$. It is finite if $\gamma < \frac{1}{2}$ (the form (26) of the graviton two-point function is assumed).

We can make the same estimates for the Wick square

$$\begin{aligned}\langle : \Phi\Phi^* : (f_{\mathbf{k}}) : \Phi\Phi^* : (f_{\mathbf{k}'})\rangle &= (2\pi)^{-2D+2d} \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \int \mathrm{d}\mathbf{p}_G\, \mathrm{d}\mathbf{p}'_G \exp\left(-\frac{1}{2a}(\mathbf{p}_G + \mathbf{p}'_G - \mathbf{k})^2\right.\\ &\quad \left.- \frac{1}{2a}(\mathbf{p}_G + \mathbf{p}'_G - \mathbf{k}')^2\right) E\left[\delta(\mathbf{y}_F - \mathbf{x}_F - \mathbf{b}_F(\tau_1))\delta(\mathbf{y}_F - \mathbf{x}_F - \mathbf{b}'_F(\tau_2))\right.\\ &\quad \times \exp\left(-\frac{1}{4}\int_0^{\tau_1}\int_0^{\tau_1} p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}_F(s) - \mathbf{b}_F(s'))\, \mathrm{d}s\, \mathrm{d}s'\right.\\ &\quad - \frac{1}{4}\int_0^{\tau_2}\int_0^{\tau_2} p'_\mu p'_\sigma p'_\nu p'_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}'_F(s) - \mathbf{b}'_F(s'))\, \mathrm{d}s\, \mathrm{d}s'\\ &\quad \left.\left.- \frac{1}{2}\int_0^{\tau_1}\int_0^{\tau_2} p_\mu p_\nu p'_\rho p'_\sigma D^{\mu\nu;\sigma\rho}(\mathbf{b}_F(s) - \mathbf{b}'_F(s'))\, \mathrm{d}s\, \mathrm{d}s'\right)\right].\end{aligned} \tag{39}$$

It is clear that we can obtain lower and upper bounds applying the Jensen inequalities (33) and (34). So, for the upper bound

$$\begin{aligned}\langle : \Phi\Phi^* : (f_{\mathbf{k}}) : \Phi\Phi^* : (f_{\mathbf{k}'})\rangle &\leqslant 2(2\pi)^{-2D+2d} \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \int_0^1 \mathrm{d}s \int_0^s \mathrm{d}s' \int \mathrm{d}\mathbf{p}_G\, \mathrm{d}\mathbf{p}'_G\\ &\quad \times \exp\left(-\frac{1}{2a}(\mathbf{p}_G + \mathbf{p}'_G - \mathbf{k})^2 - \frac{1}{2a}(\mathbf{p}_G + \mathbf{p}'_G - \mathbf{k}')^2\right)\\ &\quad \times E\left[\delta(\mathbf{y}_F - \mathbf{x}_F - \sqrt{\tau_1}\,\mathbf{b}_F(1))\delta(\mathbf{y}_F - \mathbf{x}_F - \sqrt{\tau_2}\,\mathbf{b}'_F(1))\right.\\ &\quad \times \exp\left(-\frac{1}{4}\tau_1^2 p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}(\sqrt{\tau_1}\,\mathbf{b}_F(s) - \sqrt{\tau_1}\,\mathbf{b}_F(s'))\right.\\ &\quad - \frac{1}{4}\tau_2^2 p'_\mu p'_\sigma p'_\nu p'_\rho D^{\mu\nu;\sigma\rho}(\sqrt{\tau_2}\,\mathbf{b}'_F(s) - \sqrt{\tau_2}\,\mathbf{b}'_F(s'))\\ &\quad \left.\left.- \frac{1}{2}\tau_1\tau_2 p_\mu p_\nu p'_\rho p'_\sigma D^{\mu\nu;\sigma\rho}(\sqrt{\tau_1}\,\mathbf{b}_F(s) - \sqrt{\tau_2}\,\mathbf{b}'_F(s'))\right)\right]\end{aligned} \tag{40}$$

where the rhs of equation (40) can be expressed by the transition function for the Brownian motion as in equation (35).

For the lower bound we obtain

$$\begin{aligned}\langle : \Phi\Phi^* : (f_{\mathbf{k}}) : \Phi\Phi^* : (f_{\mathbf{k}'})\rangle &\geqslant (2\pi)^{-2D+2d} \int \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \int \mathrm{d}\mathbf{p}_G\, \mathrm{d}\mathbf{p}'_G \tau_1^{-\frac{d}{2}} \tau_2^{-\frac{d}{2}} \exp\left(-\frac{1}{2a}(\mathbf{p}_G\right.\\ &\quad \left.+ \mathbf{p}'_G - \mathbf{k})^2 - \frac{1}{2a}(\mathbf{p}_G + \mathbf{p}'_G - \mathbf{k}')^2\right) \exp\left(-E\left[\frac{1}{4}\int_0^1\int_0^1 p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}\right.\right.\\ &\quad \times (\mathbf{a}(s) - \mathbf{a}(s'))\, \mathrm{d}s\, \mathrm{d}s' - \frac{1}{4}\int_0^1\int_0^1 p'_\mu p'_\sigma p'_\nu p'_\rho D^{\mu\nu;\sigma\rho}(\mathbf{a}'(s) - \mathbf{a}'(s'))\, \mathrm{d}s\, \mathrm{d}s'\\ &\quad \left.\left.- \frac{1}{2}\int_0^1\int_0^1 p_\mu p_\nu p'_\rho p'_\sigma D^{\mu\nu;\sigma\rho}(\mathbf{a}(s) - \mathbf{a}'(s'))\, \mathrm{d}s\, \mathrm{d}s'\right]\right)\end{aligned} \tag{41}$$

where we denoted

$$\mathbf{a}(s) = \mathbf{x}_F + (\mathbf{y}_F - \mathbf{x}_F)s + \sqrt{\tau_1}\,\mathbf{c}(s)$$

and

$$\mathbf{a}'(s) = \mathbf{x}_F + (\mathbf{y}_F - \mathbf{x}_F)s + \sqrt{\tau_2}\,\mathbf{c}'(s).$$

The rhs of equation (41) can be calculated explicitly using the correlation function for the Brownian bridge.

We compute now higher order correlation functions

$$\begin{aligned}
\langle\Phi(x)\Phi(x')\Phi^*(y)\Phi^*(y')\rangle &= \langle\mathcal{A}^{-1}(x,y)\mathcal{A}^{-1}(x',y')\rangle + (x \to x') \\
&= (2\pi)^{-2D+2d}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 \int \mathrm{d}\mathbf{p}_G\,\mathrm{d}\mathbf{p}'_G \exp(\mathrm{i}\mathbf{p}_G(\mathbf{y}_G - \mathbf{x}_G) + \mathrm{i}\mathbf{p}'_G(\mathbf{y}'_G - \mathbf{x}'_G)) \\
&\quad \times E\Big[\delta(\mathbf{y}_F - \mathbf{x}_F - \mathbf{b}_F(\tau_1))\delta(\mathbf{y}'_F - \mathbf{x}'_F - \mathbf{b}'_F(\tau_2)) \\
&\quad \times \exp\Big(-\frac{1}{4}\int_0^{\tau_1}\int_0^{\tau_1} p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}_F(s) - \mathbf{b}_F(s'))\,\mathrm{d}s\,\mathrm{d}s' \\
&\quad - \frac{1}{4}\int_0^{\tau_2}\int_0^{\tau_2} p'_\mu p'_\sigma p'_\nu p'_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}'_F(s) - \mathbf{b}'_F(s'))\,\mathrm{d}s\,\mathrm{d}s' \\
&\quad - \frac{1}{2}\int_0^{\tau_1}\int_0^{\tau_2} p_\mu p_\nu p'_\rho p'_\sigma D^{\mu\nu;\sigma\rho}(\mathbf{x}_F - \mathbf{x}'_F + \mathbf{b}_F(s) \\
&\quad - \mathbf{b}'_F(s'))\,\mathrm{d}s\,\mathrm{d}s'\Big)\Big] + (x \to x')
\end{aligned} \tag{42}$$

where $(x \to x')$ means the same expression in which x is exchanged with x'. The four-linear form (42) calculated on the basis f reads

$$\begin{aligned}
\left\langle\Phi\left(f_{\mathbf{k}_1}\right)\Phi\left(f_{\mathbf{k}_3}\right)\Phi^*\left(f_{\mathbf{k}_2}\right)\Phi^*\left(f_{\mathbf{k}_4}\right)\right\rangle &= (2\pi)^{-2D+2d}\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2 \int \mathrm{d}\mathbf{p}_G\,\mathrm{d}\mathbf{p}'_G E\Big[\delta(\mathbf{y}_F - \mathbf{x}_F \\
&\quad - \mathbf{b}_F(\tau_1))\delta(\mathbf{y}'_F - \mathbf{x}'_F - \mathbf{b}'_F(\tau_2)) \exp\Big(-\frac{1}{2a}(\mathbf{p}_G - \mathbf{k}_1)^2 - \frac{1}{2a}(\mathbf{p}_G - \mathbf{k}_2)^2 \\
&\quad - \frac{1}{2a}(\mathbf{p}'_G - \mathbf{k}_3)^2 - \frac{1}{2a}(\mathbf{p}'_G - \mathbf{k}_4)^2\Big) \exp\Big(-\frac{1}{4}\int_0^{\tau_1}\int_0^{\tau_1} p_\mu p_\sigma p_\nu p_\rho D^{\mu\nu;\sigma\rho} \\
&\quad \times (\mathbf{b}_F(s) - \mathbf{b}_F(s'))\,\mathrm{d}s\,\mathrm{d}s' - \frac{1}{4}\int_0^{\tau_2}\int_0^{\tau_2} p'_\mu p'_\sigma p'_\nu p'_\rho D^{\mu\nu;\sigma\rho}(\mathbf{b}'_F(s) \\
&\quad - \mathbf{b}'_F(s'))\,\mathrm{d}s\,\mathrm{d}s' - \frac{1}{2}\int_0^{\tau_1}\int_0^{\tau_2} p_\mu p_\nu p'_\rho p'_\sigma D^{\mu\nu;\sigma\rho}(\mathbf{x}_F - \mathbf{x}'_F + \mathbf{b}_F(s) \\
&\quad - \mathbf{b}'_F(s'))\mathrm{d}s\,\mathrm{d}s'\Big)\Big] + (1,2 \to 3,4)
\end{aligned} \tag{43}$$

where the last term means the same expression with exchanged wave numbers. We introduce the spherical coordinates on the (τ_1, τ_2)-plane where $\tau_1 = r\cos\theta$ and $\tau_2 = r\sin\theta$. Let us rescale the momenta $\mathbf{k} = \mathbf{p}\sqrt{r}$, $\mathbf{k}' = \mathbf{p}'\sqrt{r}$, $\mathbf{k} = \mathbf{p}r^{\frac{1}{2}-\frac{\gamma}{2}}$ and $\mathbf{k}' = \mathbf{p}'_G r^{\frac{1}{2}-\frac{\gamma}{2}}$. Then, we can see that the four-point function (42) takes the form

$$\begin{aligned}
\langle\Phi(x)\Phi(x')\Phi^*(y)\Phi^*(y')\rangle &= \int \mathrm{d}\theta\,\mathrm{d}r\, r r^{-d-(1-\gamma)(D-d)} F_4\big(\theta, r^{-\frac{1}{2}}(\mathbf{x}_F - \mathbf{y}_F), r^{-\frac{1}{2}}(\mathbf{x}'_F - \mathbf{y}'_F), \\
&\quad r^{-\frac{1}{2}}(\mathbf{x}'_F - \mathbf{x}_F), r^{-\frac{1}{2}}(\mathbf{x}'_F - \mathbf{y}_F), r^{-\frac{1}{2}}(\mathbf{y}'_F - \mathbf{x}), r^{-\frac{1}{2}+\frac{\gamma}{2}}(\mathbf{x}_G - \mathbf{y}_G), \\
&\quad r^{-\frac{1}{2}+\frac{\gamma}{2}}(\mathbf{x}'_G - \mathbf{y}'_G), r^{-\frac{1}{2}+\frac{\gamma}{2}}(\mathbf{x}'_G - \mathbf{y}_G), r^{-\frac{1}{2}+\frac{\gamma}{2}}(\mathbf{x}_G - \mathbf{y}'_G)\big).
\end{aligned} \tag{44}$$

We can conclude from equation (44) just by scaling that the singularity of the four-point function is a product of singularities.

We must now define the $\int(\Phi\Phi^*)^2$ interaction. First, let us calculate the two-point function of the interaction Lagrangian

$$\langle:(\Phi\Phi^*)^2:(x):(\Phi\Phi^*)^2:(y)\rangle = \langle(\mathcal{A}^{-1}(x,y))^4\rangle$$
$$= (2\pi)^{-2D+2d}\int E\left[\prod_{a=1}^{a=4} \mathrm{d}\tau_a\,\mathrm{d}\mathbf{p}^a \exp(\mathrm{i}\mathbf{p}^a(\mathbf{y}_G-\mathbf{x}_G))\delta\left(\mathbf{y}_F-\mathbf{x}_F-\mathbf{b}_F^a(\tau_a)\right)\right.$$
$$\left.\times\exp\left(-\frac{1}{4}\sum_{a,a'}\int_0^{\tau_a}\int_0^{\tau_{a'}} p_\mu^a p_\sigma^a p_\nu^{a'} p_\rho^{a'} D^{\mu\nu;\sigma\rho}\left(\mathbf{b}_F^a(s)-\mathbf{b}_F^{a'}(s')\right)\mathrm{d}s\,\mathrm{d}s'\right)\right]. \tag{45}$$

For the lower bound it is sufficient if we let $\mathbf{x}_G = \mathbf{y}_G$. Then

$$\langle:(\Phi\Phi^*)^2:(x):(\Phi\Phi^*)^2:(y)\rangle \geqslant (2\pi)^{-2D+2d}\int\prod_{a=1}^{a=4}\mathrm{d}\tau_a\,\mathrm{d}\mathbf{p}^a\,\tau_a^{-\frac{d}{2}}$$
$$\times\exp\left(-\frac{1}{4}E\left[\sum_{a,a'}\int_0^1\int_0^1 p_\mu^a p_\sigma^a p_\nu^{a'} p_\rho^{a'} D^{\mu\nu;\sigma\rho}\left(\mathbf{a}_F^a(s)-\mathbf{a}_F^{a'}(s')\right)\mathrm{d}s\,\mathrm{d}s'\right]\right) \tag{46}$$

where the expectation value in the exponential can be calculated and the integral on the rhs is finite. For the upper bound of the interaction $\int_V \mathrm{d}x(\Phi\Phi^*)^2(x)$ we may take the test functions of equation (32) with $2\pi a = V^{-\frac{2}{D-d}}$ and $\mathbf{k}=0$ in order to approximate the finite volume integral. With these test functions we obtain the upper bounds in the same way as we did in equations (35) and (40) from the Jensen inequality (34).

Through an introduction of spherical coordinates in the $(\tau_1,\ldots,\tau_4)$ space we can show that for short distances

$$\langle:(\Phi\Phi^*)^2:(x):(\Phi\Phi^*)^2:(y)\rangle \simeq (\langle\mathcal{A}^{-1}(x,y)\rangle)^4 \tag{47}$$

(because the r-integral scales as four times the τ-integral in equation (28)).

Let us now calculate the vacuum diagram

$$I_2 = \int_V \mathrm{d}x\int_V \mathrm{d}y\langle:(\Phi\Phi^*)^2:(x):(\Phi\Phi^*)^2:(y)\rangle \tag{48}$$

corresponding to the second-order perturbation expansion in the coupling constant. In the momentum space the convergence of this diagram follows from the convergence of the integral (for large momenta)

$$\int \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\langle\mathcal{A}^{-1}(k_1)\mathcal{A}^{-1}(k_2)\mathcal{A}^{-1}(k_3)\mathcal{A}^{-1}(k_1+k_2+k_3)\rangle. \tag{49}$$

On the basis of equation (47) the integral (49) can be approximated by

$$\int \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3\langle\mathcal{A}^{-1}(k_1)\rangle\langle\mathcal{A}^{-1}(k_2)\rangle\langle\mathcal{A}^{-1}(k_3)\rangle\langle\mathcal{A}^{-1}(k_1+k_2+k_3)\rangle. \tag{50}$$

In the proper time representation the convergence of the integral (50) depends on the convergence of the integral (for small τ)

$$\int \mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}\tau_3\,\mathrm{d}\tau_4(\tau_1+\tau_4)^{\frac{d}{2}}\left(-\tau_1^2\tau_4^2+(\tau_1\tau_3+\tau_1\tau_4+\tau_3\tau_4)(\tau_1\tau_2+\tau_1\tau_4+\tau_2\tau_4)\right)^{-\frac{d}{2}}$$
$$\times\left(\tau_1^{1-\gamma}+\tau_4^{1-\gamma}\right)^{-\frac{d}{2}+\frac{D}{2}}\left(-\tau_1^{2-2\gamma}\tau_4^{2-2\gamma}+\left(\tau_1^{1-\gamma}\tau_3^{1-\gamma}+\tau_1^{1-\gamma}\tau_4^{1-\gamma}+\tau_3^{1-\gamma}\tau_4^{1-\gamma}\right)\right.$$
$$\left.\times\left(\tau_1^{1-\gamma}\tau_2^{1-\gamma}+\tau_1^{1-\gamma}\tau_4^{1-\gamma}+\tau_2^{1-\gamma}\tau_4^{1-\gamma}\right)\right)^{\frac{d}{2}-\frac{D}{2}}. \tag{51}$$

We obtained formula (51) from representation (19) by scaling the momenta as in the argument at the end of section 3 (we could obtain lower and upper bounds on the expectation value (48) using the lower and upper bounds on the Wick powers as in equations (40) and (46); however the explicit integrals are harder for analysis than equation (51)). In order to investigate the convergence of the integral (51) it is useful to introduce the spherical coordinates. Then, the condition for the convergence of the radial part of the integral (51) reads

$$\tfrac{3}{2}d+\tfrac{3}{2}(D-d)(1-\gamma)<4. \tag{52}$$

For $D=4$ this condition has a solution only if $d=1$ and $\gamma>\frac{4}{9}$ (together with $\gamma<\frac{1}{2}$). If condition (52) is satisfied then we obtain quantum field theory without any divergencies. However, with any $\gamma>0$ the ultraviolet behaviour is better than that with $\gamma=0$. As can be deduced from the dimensional regularization, if a diagram is only logarithmically divergent (as for the charge renormalization in the $D=4$ model), then after coupling to quantum gravity it becomes convergent. In $D=4$ with $\gamma>0$ (but without the inequality (52)) the vacuum diagrams are divergent but those corresponding to the charge renormalization are convergent. We calculate, e.g., the four-point function in perturbation expansion at the second order in the coupling constant

$$\int \mathrm{d}^4z_1\,\mathrm{d}^4z_2\langle\Phi(x_1)\Phi(x_2)\Phi^*(y_1)\Phi^*(y_2):(\Phi\Phi^*)^2:(z_1):(\Phi\Phi^*)^2:(z_2)\rangle. \tag{53}$$

It is clear from our estimates that the four-point function (53) will be non-trivial and finite with any $\gamma>0$. We can calculate n-point functions. It follows that with the lower and upper bounds established in this section we can obtain finite lower and upper bounds on each term (eventually after a mass renormalization) of the perturbation series of the $(\Phi\Phi^*)^2$ model.

The Gaussian metric with the two-point function (26) may be unphysical. We applied this model in order to obtain explicit estimates on correlation functions in the perturbation series. As shown in section 3 the regularizing property of the singular random metric is universal. We expect that with some harder work the estimates on the perturbation expansion in $(\Phi\Phi^*)^2$, with the two-dimensional gravity of section 1, are possible as well.

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